

A Condensation of Interacting Bosons in Two Dimensional Space

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Abstract

We develop a theory of non-relativistic bosons in two spatial dimensions with a weak short range attractive interaction. In the limit as the range of the interaction becomes small, there is an ultra-violet divergence in the problem. We devise a scheme to remove this divergence and produce a completely finite formulation of the theory. This involves reformulating the dynamics in terms of a new operator whose eigenvalues give the *logarithm* of the energy levels. Then, a mean field theory is developed which allows us to describe the limit of a large number of bosons. The ground state is a new kind of condensate (soliton) of bosons that breaks translation invariance spontaneously. The ground state energy is negative and its magnitude grows *exponentially* with the number of particles, rather than like a power law as for conventional many body systems.

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There is a well-known theory of Bose gases in one dimensional space with a delta-function interaction [1,2]. However the two-dimensional bose system with *attractive* delta-function interactions is logarithmically divergent even in the case of two bodies [3,4,6,7]. The hamiltonian of the theory is scale invariant so that the ground state energy is either zero or negative infinity; in the case of attractive interactions it is negative infinity. The two-body problem has been intensely studied even beyond perturbation theory and it is now understood how this divergence can be removed by a renormalization. Recently we extended this analysis to the case of the three body problem; while not exactly solvable, the problem can be reformulated in a finite form without using perturbation theory [7].

In this paper (for a more detailed discussion see Ref. [8]) we will describe a non-perturbative, fully renormalized formalism for the corresponding non-relativistic field theory; i.e., the case of an arbitrary number of bosons. We will then develop a mean field theory to obtain the ground state of the system. The results are striking: the ground state is a new kind of condensate (or soliton, a bound state of a large number of particles) of bosons, which breaks translation invariance spontaneously. Moreover, the binding energy grows exponentially with the number of particles.

Of course, such a study of the binding of bosons is only possible because we now have a non-perturbative renormalization method for this theory. This bosonic field theory is asymptotically free (i.e., the running coupling constant decreases logarithmically with energy) and hence provides a good testing for our new ideas of renormalization [8]. These ideas apply also to other asymptotically free theories such as the two-dimensional non-abelian Thirring model [9] and perhaps, even to four dimensional gauge theories such as Quantum ChromoDynamics.

We begin with the naive hamiltonian of the system of bosons with an attractive self-interaction of zero range [2]:

$$H = \frac{1}{2} \int |\nabla \hat{\phi}|^2 d^2x - g \int \hat{\phi}^{\dagger 2}(x) \hat{\phi}^2(x) d^2x = \int \frac{p^2}{2} \phi^{\dagger}(p) \phi(p) \frac{dp}{(2\pi)^2}$$

$$(1) \quad -g \int \frac{dp_1 dp_2 dp'_1 dp'_2}{(2\pi)^8} (2\pi)^2 \delta(p_1 + p_2 - p'_1 - p'_2) \phi^\dagger(p_1) \phi^\dagger(p_2) \phi(p'_1) \phi(p'_2).$$

Here, $\hat{\phi}(x)$ is a complex scalar field in two dimensional space satisfying the usual canonical commutation relations; $\phi(p) = \int e^{-ip \cdot x} \hat{\phi}(x) dx$ is its Fourier transform. In the units we are using, $\hat{\phi}$ has dimensions of $(\text{length})^{-1}$ and H has dimensions of $(\text{length})^{-2}$; it follows that the coupling constant g is dimensionless. This means that the lowest eigenvalue of H (in a sector with fixed number of particles) is either zero or infinite. If $g > 0$, corresponding to an attractive potential, it is known that the ground state energy is $-\infty$ even in the case of two particles [6].

Following the standard ideas of field theory, we must regularize the problem by, for example, introducing a cut-off in the momentum; also, the coupling constant must be made a function of the cut-off in such a way that the ground state energy is finite. This is not always possible; for example the analogous problem in three dimensions requires an infinite number of coupling constants to be adjusted this way. When it is possible to remove all the divergences, by making a finite number of coupling constants depend on the cut-off, we say that the theory is renormalizable. We will show that our bosonic field theory in two dimensions is renormalizable. Moreover, we will produce a completely finite reformulation of the theory *without the use of perturbation methods*. This allows us to ask questions about the bound states of the system, even the bound state of a large number of particles. a more detailed account of the methods used is in Ref. [8].

The regularized hamiltonian is, in momentum space,

$$H_\Lambda = H_0 + H_{1\Lambda} \tag{2}$$

where

$$H_0 = \int \frac{p^2}{2} \tilde{\phi}^\dagger(p) \tilde{\phi}(p) \frac{dp}{(2\pi)^2} \tag{3}$$

and

$$H_{1\Lambda} = -g(\Lambda) \int \frac{dp_1 dp_2 dp'_1 dp'_2}{(2\pi)^8} \rho_\Lambda(p_1 - p_2) \rho_\Lambda(p'_1 - p'_2) (2\pi)^2 \delta(p_1 + p_2 - p'_1 - p'_2) \phi^\dagger(p_1) \phi^\dagger(p_2) \phi(p'_1) \phi(p'_2). \quad (4)$$

Here ρ_Λ is a cut-off function that is equal to one near the origin in momentum space and vanishes rapidly at infinity. For example we could choose

$$\rho_\Lambda(p) = \theta(|p| < \Lambda). \quad (5)$$

We will see that this cut-off in the relative momentum is sufficient to regularize the theory.

The solution of a quantum mechanical system is equivalent to determining the resolvent (Green's function) of its hamiltonian. Thus we would have an exact renormalization of our theory if we can construct the resolvent $(H_\Lambda - E)^{-1}$ in the limit as $\Lambda \rightarrow \infty$. Our key result is that such a formula for the resolvent can be obtained,

$$\begin{aligned} \lim_{\Lambda \rightarrow \infty} \frac{1}{H_\Lambda - E} &:= R(E) \\ &= \frac{1}{H_0 - E} + \frac{1}{2} \frac{1}{H_0 - E} b^\dagger \left[\frac{1}{2\pi} \log \frac{-E}{\mu^2} + W \right]^{-1} b \frac{1}{H_0 - E}. \end{aligned} \quad (6)$$

We will give below explicit formulas for the operators b and W . This gives an indirect construction of the hamiltonian of the renormalized theory as the operator for which $R(E) = [H - E]^{-1}$. We cannot get an explicit formula for the hamiltonian itself; instead we express its resolvent in terms of the resolvent of an explicitly known operator, W . The dynamical information about the system is thus encoded in the integral operator W ('the principal operator'); it plays as important a role in our theory as the hamiltonian does in usual physical theories. Once we reformulate the theory this way, the standard approximation methods of quantum theory can be applied to the principal operator. Later on we will in fact apply mean field methods to get the ground state energy of the many body system.

The operators H_0 , b , W involve no dimensional parameters; the only parameter in the theory is μ . For example the bound state energy levels of the system are given by the eigenvalues w_a of W :

$$E_a = -\mu^2 e^{-2\pi w_a}. \quad (7)$$

Once we determine one eigenvalue, it fixes μ and then all other eigenvalues are determined. For example, $-\mu^2$ is the ground state energy of the two-body system; once we know that the ground state of the n -body system for any n is determined. The original dimensionless coupling constant has been traded for the dimensional constant μ in the process of renormalization: this is dimensional transmutation [10].

In order to get the above formula for the resolvent, we have to overcome some technical difficulties. The coupling constant appears in H_Λ multiplicatively, which makes it difficult to perform a renormalization except, of course, in perturbation theory. We will introduce some auxilliary variables that will help us to rewrite the theory in such a way that coupling constant appears additively rather multiplicatively. This is key to our method for exact renormalization. Define creation-annihilation operators $\tilde{\chi}(p), \tilde{\chi}^\dagger(p)$ satisfying the product (*not commutation* relations)

$$\tilde{\chi}(p)\tilde{\chi}^\dagger(p') = (2\pi)^2\delta(p-p'), \quad \tilde{\chi}(p)\tilde{\chi}(p') = 0 = \tilde{\chi}^\dagger(p)\tilde{\chi}^\dagger(p'). \quad (8)$$

In addition, these operators commute with the bosonic fields. These relations are the $q \rightarrow 0$ limit of the q -oscillators studied by many authors (e.g., [11]). These relations have an obvious representation on $C \oplus L^2(R^2)$: there is an empty state $|0\rangle$ and states containing one auxilliary particle such as $\int v(p)\tilde{\chi}^\dagger(p)\frac{dp}{(2\pi)^2}|0\rangle$. Since the product of the creation operators vanish, there can no more than one such particle in *any* state: far more restrictive than the Pauli exclusion principle for fermions. We don't have to attach a physical significance to these new auxilliary particles: they can be viewed as a mathematical device that simplifies the analysis. ¹

¹However there are examples (quarks or ghosts) of such auxilliary variables that are important physically as well. We propose that the auxilliary particles created by our operators χ, χ^\dagger be called angels.

On the combined Hilbert space \mathcal{B}' of bosons and the auxilliary particles ($\mathcal{B}' = \mathcal{B} \oplus \mathcal{B} \otimes L^2(R^2)$, where \mathcal{B} is the usual bosonic Fock space), we define the operator:

$$H'_\Lambda = H_0 \Pi_0 + \int [dp_1 dp_2 dp_3] \rho_\Lambda(p_1 - p_2) \phi^\dagger(p_1) \phi^\dagger(p_2) \chi(p_3) (2\pi)^d \delta(p_1 + p_2 - p_3) + h.c. \Big] + \frac{1}{g(\Lambda)} \Pi_1. \quad (9)$$

Here, Π_0, Π_1 are the projection operators to the subspaces containing zero or one auxilliary particle respectively:

$$\Pi_0 = \int [dp] \chi(p) \chi^\dagger(p), \quad \Pi_1 = \int [dp] \chi^\dagger(p) \chi(p). \quad (10)$$

We are interested in getting a formula for the resolvent of the hamiltonian H_Λ . We will find it convenient to think in terms of the auxilliary hamiltonian H'_Λ and the related operator $R'_\Lambda = (H'_\Lambda - E \Pi_0)^{-1}$.

Let us split the operators into 2×2 blocks according to auxilliary particle number:

$$H'_\Lambda - E \Pi_0 = \begin{pmatrix} a & b^\dagger \\ b & d \end{pmatrix}, \quad R'_\Lambda(E) = \frac{1}{\tilde{H}_\Lambda - E \Pi_0} = \begin{pmatrix} \alpha & \beta^\dagger \\ \beta & \delta \end{pmatrix}. \quad (11)$$

with

$$a, \alpha : \mathcal{B} \rightarrow \mathcal{B}, \quad b^\dagger, \beta^\dagger : \mathcal{B} \otimes L^2(R^d) \rightarrow \mathcal{B}, \quad d, \delta : \mathcal{B} \otimes L^2(R^d) \rightarrow \mathcal{B} \otimes L^2(R^d). \quad (12)$$

Two different ways of writing the elements of the inverse matrix will be useful (the proof is elementary):

$$\left[a - b^\dagger d^{-1} b \right]^{-1} = \alpha = a^{-1} + a^{-1} b^\dagger [d - b a^{-1} b^\dagger]^{-1} b a^{-1}, \quad (13)$$

$$\left[d - b a^{-1} b^\dagger \right]^{-1} = \delta = d^{-1} + d^{-1} b [a - b^\dagger d^{-1} b]^{-1} b^\dagger d^{-1}, \quad (14)$$

$$-\delta b a^{-1} = \beta = -d^{-1} b \alpha. \quad (15)$$

Using the left hand side of the formula for α we can see that

$$\alpha = (H_\Lambda - E)^{-1} \quad (16)$$

is the just the resolvent of the original bosonic system. The other way to write this quantity gives us

$$\frac{1}{H_\Lambda - E} = \frac{1}{H_0 - E} + \frac{1}{2} \frac{1}{H_0 - E} b^\dagger \Phi_\Lambda(E)^{-1} b \frac{1}{H_0 - E}. \quad (17)$$

Here

$$\begin{aligned} \Phi_\Lambda(E) = & \frac{1}{g(\Lambda)} - \int \frac{dp_1 dp_2 dp'_1 dp'_2}{(2\pi)^8} \rho_\Lambda(p_1 - p_2) \rho_\Lambda(p'_1 - p'_2) \\ & \chi^\dagger(p_1 + p_2) \left[\phi(p_1) \phi(p_2) \frac{1}{H_0 - E} \phi^\dagger(p'_1) \phi^\dagger(p'_2) \right] \chi(p'_1 + p'_2). \end{aligned} \quad (18)$$

and

$$b^\dagger = \int [dp_1 dp_2] \rho_\Lambda(p_1 - p_2) \phi^\dagger(p_1) \phi^\dagger(p_2) \chi(p_1 + p_2). \quad (19)$$

The point of this formula is that it expresses the resolvent of the hamiltonian in terms of that of the free theory and the operator $\Phi_\Lambda(E)$. Moreover, the coupling constant appears additively in Φ_Λ ; we will be able to separate the divergence in Φ_Λ and remove it by choosing $g(\Lambda)$ to be an appropriate function of Λ . Then the limiting operator $\Phi(E) = \lim_{\Lambda \rightarrow \infty} \Phi_\Lambda(E)$ will exist. The roots of the equation $\Phi(E)|s\rangle = 0$ will give the energy levels. Thus we will obtain a completely finite reformulation of the problem: what we have called elsewhere a transfinite formulation. It is important that no where in this argument we will use perturbation theory.

In more detail, we can use the canonical commutation relations to normal order the operators in Φ_Λ . This will give

$$\begin{aligned} \Phi_\Lambda(E) = & g^{-1}(\Lambda) - \int [dp_1 dp_2 dp'_1 dp'_2] \rho_\Lambda(p_1 - p_2) \rho_\Lambda(p'_1 - p'_2) \\ & \chi^\dagger(p_1 + p_2) \left[\phi^\dagger(p'_1) \phi^\dagger(p'_2) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_1) + \omega(p_2) - E} \phi(p_1) \phi(p_2) \right. \\ & + 4(2\pi)^d \delta(p_1 - p'_1) \phi^\dagger(p'_2) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_2) - E} \phi(p_2) \\ & \left. + 2(2\pi)^d \delta(p_1 - p'_1) (2\pi)^d \delta(p_2 - p'_2) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) - E} \right] \\ & \chi(p'_1 + p'_2). \end{aligned} \quad (20)$$

The only divergent term is the first one the square brackets. The divergence is removed if we choose

$$g^{-1}(\Lambda) = \int \rho_\Lambda^2(p) \frac{1}{p^2 + \mu^2}. \quad (21)$$

Here we are trading the dimensionless coupling constant g for the dimensional constant μ . The renormalized theory will depend on μ alone and not the cut-off Λ or on the coupling constant: dimensional transmutation.

The operator $\Phi_\Lambda(E)$ has a limit as $\Lambda \rightarrow \infty$:

$$\begin{aligned} \Phi(E) = & \int [dp] \chi^\dagger(p) \xi\left(\frac{\mu^2}{2}, H_0 + \frac{p^2}{2} - E\right) \chi(p) \\ & - \int [dp_1 dp_2 dp'_1 dp'_2] \chi^\dagger(p_1 + p_2) \left[\right. \\ & \phi^\dagger(p'_1) \phi^\dagger(p'_2) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_1) + \omega(p_2) - E} \phi(p_1) \phi(p_2) \\ & \left. + 4(2\pi)^d \delta(p_1 - p'_1) \phi^\dagger(p'_2) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_2) - E} \phi(p_2) \right] \\ & \chi(p'_1 + p'_2). \end{aligned} \quad (22)$$

Here,

$$\begin{aligned} \xi\left(\frac{\mu^2}{2}, \frac{\nu^2}{2}\right) &:= \int [dp] \left[\frac{1}{p^2 + \mu^2} - \frac{1}{p^2 + \nu^2} \right] \\ &= \frac{1}{4\pi} \ln \frac{\nu^2}{\mu^2} \end{aligned} \quad (23)$$

is a convergent integral. We can rescale all the momenta by $\sqrt{|E|}$ to get ²

$$\Phi(\mu, E) = \left[\frac{1}{2\pi} \ln \frac{-E}{\mu^2} + W \right]. \quad (24)$$

(From this point on our momentum variables p, p' and position variables x are *dimensionless*.) The principal operator W is, ($\omega(p) = \frac{p^2}{2}$)

$$\begin{aligned} W = & \frac{1}{2\pi} \int [dp] \chi^\dagger(p) \log \left[H_0 + \omega(p) + 1 \right] \chi(p) \\ & - \int [dp_1 dp_2 dp'_1 dp'_2] \chi^\dagger(p_1 + p_2) \left[\right. \\ & \phi^\dagger(p'_1) \phi^\dagger(p'_2) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_1) + \omega(p_2) + 1} \phi(p_1) \phi(p_2) \\ & \left. + 4(2\pi)^d \delta(p_1 - p'_1) \phi^\dagger(p'_2) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_2) + 1} \phi(p_2) \right] \\ & \chi(p'_1 + p'_2). \end{aligned} \quad (25)$$

²We assume that $E < 0$ so that it is not in the spectrum of H_0 . The resolvent is defined elsewhere by analytic continuation.

This proves the formula (6) for the resolvent.

Thus the eigenvalues of the operator W will determine the energy levels.

In fact,

$$W|\psi\rangle = w|\psi\rangle, \quad E = -\mu^2 e^{-2\pi w}. \quad (26)$$

The problem is completely well-posed in terms of the operator W ; it is a replacement for the hamiltonian of the theory. In fact there are many subtleties in the definition of the hamiltonian itself (e.g., domain self-adjointness), all of which can be avoided by working with W . We call this remarkable new operator that embodies the dynamics of the system the ‘Principal Operator’. Roughly speaking it represents the logarithm of energy.

Now that we have a finite formulation of the theory, we can look for analogues of all the approximation methods of quantum theory. In particular, we can look for a mean field theory. The idea is to use a variational principle to estimate the lowest eigenvalue of W in the sector with n bosons. Now W acts in the Hilbert space of n bosons and one auxilliary particle. The variational ansatz corresponding to mean field theory is the one where all the bosons are in the state u and the angel is in the state v :

$$|u, v\rangle = \int \frac{dp_1 \cdots dp_n dp}{(2\pi)^{2n+2}} u(p_1) \cdots u(p_n) v(p) \phi^\dagger(p_1) \cdots \phi^\dagger(p_n) \chi^\dagger(p) |0\rangle. \quad (27)$$

The expectation value of W in this state becomes (for large n) the ‘principal function’

$$\begin{aligned} U = & \frac{1}{2\pi} \int [dp] |v(p)|^2 \log[nh_0(u) + \omega(p) + 1] \\ & - \int [dp_1 dp_2 dp'_1 dp'_2] v(p_1 + p_2)^* v(p'_1 + p'_2) \\ & \left[\frac{u^*(p'_1) u^*(p'_2) u(p_1) u(p_2) n(n-1)}{nh_0(u) + \omega(p'_1) + \omega(p'_2) + \omega(p_1) + \omega(p_2) + 1} \right. \\ & \left. + 4 \frac{(2\pi)^2 \delta(p_1 - p'_1) n u^*(p'_2) u(p_2)}{nh_0(u) + \omega(p'_1) + \omega(p'_2) + \omega(p_2) + 1} \right] \end{aligned} \quad (28)$$

where

$$h_0(u) = \int |u(p)|^2 \omega(p) [dp]. \quad (29)$$

This U is to be minimized subject to the normalization conditions

$$\int |u(p)|^2 [dp] = \int |v(p)|^2 [dp] = 1. \quad (30)$$

If we keep just the leading terms as $n \rightarrow \infty$,

$$U = -n \frac{\left| \int v^*(p_1 + p_2) u(p_1) u(p_2) [dp_1 dp_2] \right|^2}{h_0(u)} + \frac{1}{2\pi} \log n - \frac{1}{2\pi} \log f_1(u) + O(n^{-1}). \quad (31)$$

The functional $f_1(u)$ (which is independent of n) can also be determined as above, but it takes a bit more work. Thus we have the approximate variational principle for the ground state energy of a system of n bosons:

$$E_n = -\mu^2 \frac{e^{\frac{2\pi n}{\xi}}}{n} [C_1 + O(\frac{1}{n})] \quad (32)$$

where

$$\xi = \inf_{u,v} \frac{\int |v(p)|^2 [dp] \int |u(p)|^2 [dp] \int \omega(p) |u(p)|^2 [dp]}{\left| \int v^*(p_1 + p_2) u(p_1) u(p_2) [dp_1 dp_2] \right|^2}. \quad (33)$$

Also, C_1 is a constant; it is not necessary to determine it for the leading behavior in the large n limit of energy. We can eliminate v rather easily to get an equivalent form (written in terms of the position space wavefunction $\tilde{u}(x) = \int u(p) e^{ipx} \frac{dp}{(2\pi)^2}$):

$$\xi = \inf_{\tilde{u}} I[\tilde{u}] \quad (34)$$

where,

$$I[\tilde{u}] = \frac{\int |\tilde{u}(x)|^2 dx \int \frac{1}{2} |\nabla \tilde{u}(x)|^2 dx}{\int |\tilde{u}(x)|^4 dx}. \quad (35)$$

All the variables in our problem are dimensionless. The minimization of $\log I[u]$ gives the partial differential equation

$$\nabla^2 \tilde{u} - \beta \tilde{u} + \lambda |\tilde{u}|^2 \tilde{u} = 0 \quad (36)$$

where,

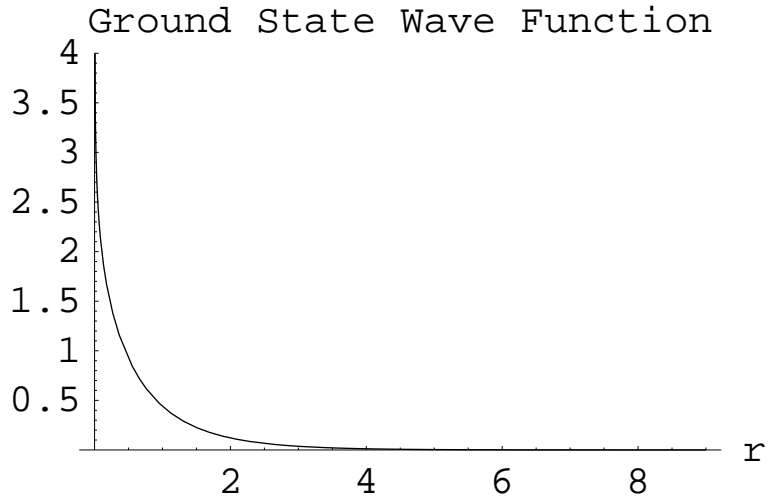
$$\beta = \frac{\int |\nabla \tilde{u}|^2 d^2x}{\int |\tilde{u}|^2 d^2x}, \quad \lambda = \frac{\int |\nabla \tilde{u}|^2 d^2x}{\int |\tilde{u}|^4 d^2x}. \quad (37)$$

This nonlinear Schrodinger equation has been shown to have solutions in [12], Theorem 6.7.25. (Because of the scale covariance it is enough to find

solutions with one positive value of β and λ .) We expect the ground state to have at least circular symmetry; we get a sort of nonlinear Bessel's equation:

$$v''(r) + \frac{1}{4r^2}v(r) + \frac{v^3(r)}{r} = v(r) \quad (38)$$

with $\tilde{u}(x) = \sqrt{rv(r)}$ and $r = |x|$. It is easy to solve this equation numerically; we plot the result in the figure. The value of ξ we get is about 12. The solution of course breaks translation invariance spontaneously but not rotation invariance. It is in fact a 'soliton' in the sense of field theory.



It follows that the ground state energy (binding energy) grows exponentially with the number of particles:

$$E_n = -\mu^2 e^{\frac{\pi}{6}n + \dots} \quad (39)$$

Here, $-\mu^2$ is the binding energy of the two boson system; the dots denote terms of lower order in the number of bosons. At zero temperature, essentially all the available particles will fall into this bound state. That is the sense in which this state is a condensate.

The wavefunction was determined above in terms of a dimensionless position variable. By putting back the factor $\sqrt{|E|}$ we scaled out, we see that the size of the condensate behaves like $|E_n|^{-\frac{1}{2}} \sim \mu^{-1} e^{-\frac{\pi}{12}n}$; it shrinks with increasing number of particles. Although the results above are asymptotic for large n , we expect (by analogy with mean field theory in atomic physics) that they will hold true even for moderate values of n of about 3 or more.

It may be possible to observe such a condensation in cold optical traps of dilute bose gases, if we can trap the atoms close to some surface [13]. Even without a confining potential in the plane of the surface, the atoms will form a ‘self-trapped’ bound state. It may be possible to tune the attractive force between atoms using Feshbach resonances [14]. Then, even with an exponentially small size for the condensate, the atoms can be far apart and the approximation of zero range will be valid. Whether large particle numbers in the condensate can be achieved is not clear yet, however. There might also be a realization in terms of vortices in superconductors which are on the boundary between type I and type II [8].

In the case of the three dimensional attractive Bose gas, even after the above renormalization the energy is not bounded below [8]; our methods fail. In fact there are indications (both theoretical and experimental) that in this case, the system is truly unstable [15,16].

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